

ASYMPTOTICS OF THE BEST POLYNOMIAL APPROXIMATION OF $|x|^p$ AND OF THE BEST LAURENT POLYNOMIAL APPROXIMATION OF $\operatorname{sgn}(x)$ ON TWO SYMMETRIC INTERVALS

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ABSTRACT. We present a new method that allows us to get a direct proof of the classical Bernstein asymptotics for the error of the best uniform polynomial approximation of $|x|^p$ on two symmetric intervals. Note, that in addition, we get asymptotics for the polynomials themselves under a certain renormalization. Also, we solve a problem on asymptotics of the best approximation of $\operatorname{sgn}(x)$ on $[-1, -a] \cup [a, 1]$ by Laurent polynomials.

1. INTRODUCTION

By S. N. Bernstein [5], [6, Ch. II, Sect. 5], see also [1], it follows that the error $E_n(p, a)$ of the best uniform approximation of $|x|^p$, p not an even integer, on $[-1, -a] \cup [a, 1]$ by polynomials of degree $n = 2m$ the following limit exists:

$$\lim_{m \rightarrow \infty} \left(\frac{1+a}{1-a} \right)^{m+1} m^{\frac{p}{2}+1} E_n(p, a) = a^{\frac{p}{2}-1} \frac{(1+a)^2}{2 \left| \Gamma\left(-\frac{p}{2}\right) \right|}. \quad (1.1)$$

Indeed, see Appendix 2, (1.1) may be derived easily from the error of the best approximation of $1/(b+x)^s$, $s \neq 0$. We mention that due to Chebyshev, see [1, chap. II, No. 37] or [5, p. 120], for $s = 1$ even the best polynomial approximation is known explicitly.

Note that for $a = 0$, that is for the approximation of $|x|^p$ on $[-1, 1]$, the asymptotics for $E_n(p) = E_n(p, 0)$ is still the open famous Bernstein problem, though the existence of the limit

$$\lim_{n \rightarrow \infty} n^p E_n(p) = \mu(p) > 0 \quad (1.2)$$

was shown in [3, 4]. In particular, one can not just put $a = 0$ in (1.1), even the growth (exponent for $n = 2m$) is different. For recent progress

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concerning the Bernstein problem, see D. Lubinsky [12, 13]. A survey on Bernstein constant theorems is given in [8], see also [9].

For results of the type (1.2), where $|x|^p$ (resp. $|x - x_0|^p$) is approximated on a system of several intervals, see [18, 16], [17, Sect. 10].

The interest to this remarkable problem (see, e.g. [12]) was boosted by the recent result of H. Stahl [15], who completed a long line of studying of the analogous problem for uniform *rational* approximation of $|x|^p$ on $[-1, 1]$ with the remarkable explicit answer:

$$\lim_{n \rightarrow \infty} \exp(\pi \sqrt{pn}) E_n^r(p) = 2^{2+p} |\sin(\pi p/2)|,$$

where $E_n^r(p)$ is the error of the best rational approximation.

E. I. Zolotarev [19, 2] found an explicit expression, in terms of elliptic functions, of the rational function of given degree which is uniformly closest to $\operatorname{sgn}(x)$ on $[-1, -a] \cup [a, 1]$.

We call a rational function of the form

$$f(x) = \frac{a_{-l}}{x^l} + \dots + a_n x^n$$

a Laurent polynomial of degree (l, n) .

Problem 1.1. For $k, m \in \mathbb{N}$, find the best approximation of the function $\operatorname{sgn}(x)$, $|x| \in [a, 1]$, by Laurent polynomials of degree $(2k - 1, 2m - 1)$ and the approximation error $L_m^k(a)$.

Remark 1.2. Problem 1.1, in a trivial way, is related to the following weighted polynomial approximation problem:

$$E_n^*(p, a) = L_m^k(a) = \inf_{\{P: \deg P \leq 2(m+k-1)\}} \sup_{|x| \in [a, 1]} \left| \frac{|x|^{2k-1} - P(x)}{x^{2k-1}} \right|, \quad (1.3)$$

where $a \in (0, 1)$, $k, m \in \mathbb{N}$ and $p = 2k - 1$, $n = 2(k + m - 1)$. Also, it is trivial that the extremal function $f = f(x; k, m; a)$ is odd in Problem 1.1 and the extremal polynomial in (1.3) is even.

Problem 1.3. For an integer m and a real p , $2m > p > 0$, find the polynomial of degree $2m$ of the best approximation to $|x|^p$, $|x|$ belongs to $[a, 1]$.

In [7] A. Eremenko and the last author solved the standard polynomial case and the Laurent case of this problem was considered in [14]. In particular, Lemma 2.1, Theorems 3.1 and 5.1, and a weak version of Theorem 4.1 were proved in preprint [14]. However, in both works, in the last step we were enforced to use Bernstein's result similar to (1.1). Here we, finally, close this approach, and, as it was mentioned before, obtain in a natural way asymptotics for the approximation error and simultaneously for the extremal function under a certain renormalization.

The main steps of the method are (with respect to Problem 1.1):

1. For each particular k and m we reveal the structure of the extremal function by representing it with the help of an explicitly given conformal mapping.

2. The system of conformal mappings (k is fixed, m is a parameter) converges (in the Caratheodory sense) after an appropriate renormalization. The limit map does not depend on a , thus we obtain asymptotics for $L_m^k(a)$ in terms of a -depending parameters, that we use for renormalization, (an explicit formula) and a k -depending constant, say Y_k , which is a certain characteristic of the final conformal map (kind of capacity).
3. Using a special representation for bounded Nevanlinna functions we get the explicit formula for the final conformal mapping, in particular, for the constant Y_k .

With a slight modification we apply the method to asymptotics related to Problem 1.3, see Sections 5 and 6. Note, that up to the last step in this problem, we can follow the same program in the most intriguing case $a = 0$. However, on the contrary to the linear equation appearing in the considered case $a > 0$, see (6.12) (or (2.12) for Problem 1.1), we get a kind of quadratic equation (7.5) involving an unknown function, its Hilbert transform and an independent variable. The trigonometric form (7.7) might be preferable for the equation. But, in any case, at the moment we are unable to find a way to get its explicit solution and for this reason we do not discuss this subject in the main part of the paper and just formulate corresponding conjecture in Appendix 1.

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2. SPECIAL CONFORMAL MAPPINGS

In this section we introduce certain special conformal mappings that we need in what follows. They are marked by a natural parameter k , but in this section k can be just real, $k > 1/2$.

For given k , consider the domain

$$\Pi_k = \mathbb{C}_+ \setminus \{w : \operatorname{Re} w = -\log t, |\operatorname{Im} w - k\pi| \leq \arccos t, t \in (0, 1]\} \quad (2.1)$$

Define the conformal map

$$H_k : \mathbb{C}_+ \rightarrow \Pi_k$$

normalized by $H_k(0) = \infty_1$, $H_k(\infty) = \infty_2$ (on the boundary we have two infinite points that we denote respectively ∞_1, ∞_2), and moreover

$$H_k(\zeta) = \zeta + \dots, \quad \zeta \rightarrow \infty,$$

(that is the leading coefficient is fixed). By D_k we denote the positive number such that $H_k(-D_k) = 0$.

Recall that a Nevanlinna function $G(z)$ ($\text{Im } G(z) > 0$, for $\text{Im } z > 0$) possesses the integral representation, see e.g. [11],

$$G(z) = Az + B + \int \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\sigma(t), \quad (2.2)$$

where $A > 0$, $B \in \mathbb{R}$, σ is a positive measure on the real axis such that $\int \frac{d\sigma(t)}{1+t^2} < \infty$. Moreover

$$A = \lim_{z=iy, y \rightarrow \infty} \frac{G(z)}{z}, \quad \sigma(x_2) - \sigma(x_1) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{x_1}^{x_2} \text{Im } G(x + i\epsilon) dx. \quad (2.3)$$

Therefore for H_k we have the following integral representation

$$H_k(\zeta) = \zeta + D_k + \int_0^\infty \left(\frac{1}{t-\zeta} - \frac{1}{t+D_k} \right) \rho_k(t) dt, \quad (2.4)$$

where $\rho_k(t) = \frac{1}{\pi} \text{Im } H_k(t)$. Evidently $\rho_k(t) \rightarrow k + \frac{1}{2}$, $t \rightarrow +\infty$.

Lemma 2.1. *The function H_k possesses the asymptotic*

$$\lim_{\zeta \rightarrow -\infty} \left\{ H_k(\zeta) - \zeta + \left(k + \frac{1}{2} \right) \log(-\zeta) \right\} = Y_k, \quad (2.5)$$

where

$$Y_k := D_k + \left(k + \frac{1}{2} \right) \log D_k - \int_0^\infty \frac{\rho_k(t) - \left(k + \frac{1}{2} \right)}{t + D_k} dt. \quad (2.6)$$

Proof. Since

$$\int_0^\infty \left(\frac{1}{t-\zeta} - \frac{1}{t+D_k} \right) \left(\rho_k(t) - \left(k + \frac{1}{2} \right) \right) dt \rightarrow - \int_0^\infty \frac{\rho_k(t) - \left(k + \frac{1}{2} \right)}{t + D_k} dt, \quad (2.7)$$

as $\zeta \rightarrow -\infty$ and

$$\left(k + \frac{1}{2} \right) \int_0^\infty \left(\frac{1}{t-\zeta} - \frac{1}{t+D_k} \right) dt = - \left(k + \frac{1}{2} \right) (\log(-\zeta) - \log D_k) \quad (2.8)$$

we get (2.5). \square

As it was mentioned in the Introduction (step 3 of our method), we will use a certain special representation for a Nevanlinna function $H_k(\zeta)$. We note that a Nevanlinna function $F(z)$ with the imaginary part in $[0, \pi]$ has the form $F(z) = \log G(z)$, where $G(z)$ is another nontrivial Nevanlinna function, see e.g. [11]. Based on this remark we get the following corollary of the previous lemma.

Corollary 2.2. *The function H_k possesses the representation*

$$H_k(\zeta) = \zeta - \left(k - \frac{1}{2} \right) \log(-\zeta) + \log \left\{ \frac{1}{\pi} \int_0^\infty \frac{\tau_k(t) dt}{t - \zeta} \right\}. \quad (2.9)$$

Proof. As it follows from (2.6) and (2.1)

$$\operatorname{Im} \left\{ H_k(\zeta) - \left(\zeta - \left(k - \frac{1}{2} \right) \log(-\zeta) \right) \right\} \in [0, \pi], \quad \operatorname{Im} \zeta > 0.$$

Using the representation (2.2) and (2.3) we get (2.9) with

$$\tau_k(x) = \arg \left\{ H_k(x) - \left(x - \left(k - \frac{1}{2} \right) \log(-x) \right) \right\}.$$

□

Theorem 2.3. *The function $H_k(z)$ is of the form*

$$H_k(\zeta) = \zeta - \left(k - \frac{1}{2} \right) \log(-\zeta) + \log \left\{ \frac{1}{\pi} \int_0^\infty \frac{t^{k-\frac{1}{2}} e^{-t} dt}{t - \zeta} \right\}. \quad (2.10)$$

in particular,

$$Y_k = \log \Gamma \left(k + \frac{1}{2} \right) - \log \pi. \quad (2.11)$$

Proof. We note that for $\zeta = \xi + i\eta$ the curve in (2.1) is given by the equation

$$\operatorname{Re}\{e^{w-i\pi k}\}|_{\zeta=\xi+i0} = 1, \quad \xi > 0.$$

We use here the representation (2.9) and, thus, get

$$\operatorname{Im} \left\{ e^\xi \xi^{-(k-\frac{1}{2})} \frac{1}{\pi} \int_0^\infty \frac{\tau_k(t) dt}{t - \zeta} \right\} \Big|_{\zeta=\xi+i0} = 1, \quad \xi > 0. \quad (2.12)$$

Therefore (2.10) is proved.

Now we use the standard asymptotic formula

$$\frac{1}{\pi} \int_0^\infty \frac{\tau_k(t) dt}{t - \zeta} = \frac{\frac{1}{\pi} \int_0^\infty \tau_k(t) dt}{-\zeta} + \frac{\frac{1}{\pi} \int_0^\infty t \tau_k(t) dt}{-\zeta^2} + \dots$$

Therefore

$$\begin{aligned} H_k(\zeta) &= \zeta - \left(k - \frac{1}{2} \right) \log(-\zeta) + \log \left\{ \frac{\frac{1}{\pi} \int_0^\infty t^{k-\frac{1}{2}} e^{-t} dt}{-\zeta} (1 + o(1)) \right\} \\ &= \zeta - \left(k + \frac{1}{2} \right) \log(-\zeta) + \log \left\{ \frac{1}{\pi} \int_0^\infty t^{k-\frac{1}{2}} e^{-t} dt \right\} + o(1). \end{aligned} \quad (2.13)$$

Due to (2.5) we get (2.11).

□

Remark 2.4. The conformal map on the domain

$$\Pi_0 = \mathbb{C}_+ \setminus \{w : \operatorname{Re} w = -\log t, \operatorname{Im} w \leq \arccos t, t \in (0, 1]\} \quad (2.14)$$

is important for a description of the standard polynomial approximation of $\operatorname{sgn}(x)$ [7]. Though it requires a special consideration, the formal extension of (2.10) to the case $k = 0$, that is, the formula

$$H_0(\zeta) = \zeta + \frac{1}{2} \log(-\zeta) + \log \left\{ \frac{1}{\pi} \int_0^\infty \frac{t^{-\frac{1}{2}} e^{-t} dt}{t - \zeta} \right\} \quad (2.15)$$

holds true for the map normalized by $H_0(0) = 0$.

3. EXTREMAL PROBLEM

For a parameter $B > 0$ and $k, m \in \mathbb{N}$, $\Omega_m^k(B)$ denotes the subdomain of the half strip

$$\{w = u + iv : v > 0, 0 < u < (k + m)\pi\}$$

that we obtain by deleting the subregion

$$\{w = u + iv : |u - \pi k| \leq \arccos\left(\frac{\cosh B}{\cosh v}\right), v \geq B\}. \quad (3.1)$$

Let $\phi(z) = \phi(z; k, m; B)$ be the conformal map of the first quadrant onto $\Omega_m^k(B)$ such that $\phi(0) = \infty_1$, $\phi(1) = (k + m)\pi$, $\phi(\infty) = \infty_2$. Let $a = \phi^{-1}(0)$. Then a is a continuous strictly increasing function of B , moreover $\lim_{B \rightarrow 0} a(B) = 0$ and $\lim_{B \rightarrow \infty} a(B) = 1$. Thus we may consider the inverse function $B(a) = B_m^k(a)$, $a \in (0, 1)$.

Theorem 3.1. *The error of the best approximation in Problem 1.1 is*

$$L_m^k(a) = \frac{1}{\cosh B_m^k(a)} \quad (3.2)$$

and the extremal function is of the form

$$f(x; k, m; a) = 1 - (-1)^k L_m^k(a) \cos \phi(x; k, m; B(a)), \quad x > 0. \quad (3.3)$$

Proof. By inspection of the boundary correspondence, we conclude that $f = 1 - (-1)^k L \cos \phi$ is real on the positive ray and pure imaginary on the positive imaginary ray. So by two reflections f extends to a function analytic in $\mathbb{C} \setminus \{0\}$. The extended function evidently satisfies

$$\overline{f(\bar{z})} = f(z) \quad \text{and} \quad -\overline{f(-\bar{z})} = f(z),$$

so we conclude that f is odd. The region $\Omega_m^k(B)$ is close to the strip

$$\left\{w : \operatorname{Re} w \in \left(0, \pi \left(k - \frac{1}{2}\right)\right)\right\}$$

as $\operatorname{Im} w \rightarrow \infty_1$, and to the strip

$$\left\{w : \operatorname{Re} w \in \left(\pi \left(k + \frac{1}{2}\right), \pi(k + m)\right)\right\}$$

as $\operatorname{Im} w \rightarrow \infty_2$. So $\phi \sim (2k - 1) \log 1/z$, $z \rightarrow 0$, $\phi \sim (2m - 1) \log z$, $z \rightarrow \infty$, and, therefore, f is a Laurent polynomial of degree $(2k - 1, 2m - 1)$. Now we note that the graph of f alternates $k + m + 1$ times on $[a, 1]$ between $1 - L$ and $1 + L$. That a Laurent polynomial with such graph is the unique extremal for Problem 1.1 follows from the general theorem of Chebyshev on the uniform approximation of continuous functions [1, Ch. II].

Finally, we have to note that on the imaginary axis the extremal function has precisely one zero (there are no critical points and the behavior at $i0$ and at $i\infty$ is evident). At this point $\phi = k\pi + iB$ and we have (3.2). \square

4. ASYMPTOTICS

Theorem 4.1. *The following limit exists*

$$\begin{aligned} \lim_{m \rightarrow \infty} \left\{ B_m^k(a) - \left(m - \frac{1}{2}\right) \log \frac{1+a}{1-a} - \left(k + \frac{1}{2}\right) \log(2m-1) \right\} \\ = \left(k + \frac{1}{2}\right) \log \frac{a}{1-a^2} - \log \frac{\Gamma(k+1/2)}{\pi}. \end{aligned} \quad (4.1)$$

Moreover, uniformly on compact subsets of the positive half-axis,

$$\begin{aligned} \lim_{m \rightarrow \infty} f \left(\sqrt{\frac{2a}{2m-1}} \lambda; k, m; a \right) \\ = 1 + \frac{(-1)^{k+1}}{\pi} \int_0^\infty \left(\frac{\mu}{\lambda} \right)^{2k-1} e^{-(\lambda^2 + \mu^2)} \frac{2\mu d\mu}{\lambda^2 + \mu^2}. \end{aligned} \quad (4.2)$$

Proof. We use the symmetry principle and make a convenient changes of variable to have a conformal map $\Phi_m(Z) = \Phi(Z; k, m; B)$ of the upper Z -plane

$$Z = C_m \sqrt{\frac{z^2 - a^2}{z^2 - 1}} \quad (4.3)$$

in the region

$$i(\Omega_m^k(B) \cup \overline{\Omega_m^k(B)}) \cup (0, i\pi(m+k)).$$

This conformal map has the following boundary correspondence

$$\Phi_m : (-C_m, -A_m, 0, A_m, C_m) \rightarrow (-\infty_2, -\infty_1, 0, \infty_1, \infty_2),$$

here $A_m = aC_m$ and the parameter C_m will be chosen a bit later.

For Φ_m we have the following integral representation

$$\Phi_m(Z) = \left(m - \frac{1}{2}\right) \log \frac{1 + \frac{Z}{C_m}}{1 - \frac{Z}{C_m}} + \int_{A_m}^\infty \left[\frac{1}{X - Z} - \frac{1}{X + Z} \right] v_m(X) dX,$$

where

$$v_m(X) = \begin{cases} \frac{1}{\pi} \text{Im} \Phi_m(X), & A_m \leq X \leq C_m \\ k + \frac{1}{2}, & X > C_m \end{cases} \quad (4.4)$$

Put now

$$H_m^k(\zeta) = \Phi_m(Z) - B_m, \quad Z = A_m + \zeta,$$

then

$$\begin{aligned} H_m^k(\zeta) = \left(m - \frac{1}{2}\right) \log \frac{1 + a + \frac{\zeta}{C_m}}{1 - a - \frac{\zeta}{C_m}} + \int_0^\infty \left[\frac{1}{t - \zeta} - \frac{1}{t + 2A_m + \zeta} \right] \hat{v}_m(t) dt \\ - B_m, \end{aligned}$$

where $\hat{v}_m(t) = v_m(t + A_m)$. Let us rewrite H_m^k in the form that is close to the integral representation of H_k :

$$\begin{aligned} H_m^k(\zeta) &= \left(m - \frac{1}{2}\right) \log \frac{1 + \frac{\zeta}{C_m(1+a)}}{1 - \frac{\zeta}{C_m(1-a)}} + D_k + \int_0^\infty \left[\frac{1}{t - \zeta} - \frac{1}{t + D_k} \right] \hat{v}_m(t) dt \\ &\quad + \left(m - \frac{1}{2}\right) \log \frac{1+a}{1-a} - D_k + \int_0^\infty \left[\frac{1}{t + D_k} - \frac{1}{t + 2A_m + \zeta} \right] \hat{v}_m(t) dt \\ &\quad - B_m \end{aligned} \quad (4.5)$$

Now, we put

$$C_m = \frac{2m-1}{1-a^2}.$$

In this case the first line in (4.5) converges to $H_k(\zeta)$. Since

$$\begin{aligned} \lim_{m \rightarrow \infty} \int_0^\infty \left[\frac{1}{t + D_k} - \frac{1}{t + 2A_m + \zeta} \right] \left(\hat{v}_m(t) - \left(k + \frac{1}{2}\right) \right) dt \\ = \int_0^\infty \frac{\rho_k(t) - (k + \frac{1}{2})}{t + D_k} dt \end{aligned} \quad (4.6)$$

and

$$\int_0^\infty \left[\frac{1}{t + D_k} - \frac{1}{t + 2A_m + \zeta} \right] dt = \log \frac{2A_m}{D_k} + \log \left(1 + \frac{\zeta}{2A_m} \right) \quad (4.7)$$

we have from the second line in (4.5) that

$$\begin{aligned} \lim_{m \rightarrow \infty} \left\{ B_m - \left(m - \frac{1}{2}\right) \log \frac{1+a}{1-a} - \left(k + \frac{1}{2}\right) \log 2A_m \right\} \\ = -D_k - \left(k + \frac{1}{2}\right) \log D_k + \int_0^\infty \frac{\rho_k(t) - (k + \frac{1}{2})}{t + D_k} dt = -Y_k. \end{aligned} \quad (4.8)$$

By (2.11) we get (4.1).

Now, let us transform the convergence of conformal mappings in the asymptotic for the extremal function. From (4.3) we have

$$z = \sqrt{\frac{Z^2 - A_m^2}{Z^2 - C_m^2}} = \sqrt{\frac{2\zeta A_m + \zeta^2}{A_m^2 - C_m^2 + 2\zeta A_m + \zeta^2}} \sim \sqrt{\frac{2a}{2m-1}} \sqrt{-\zeta}. \quad (4.9)$$

From (3.3) we get the following chain of equalities for the depending variable

$$\begin{aligned} f(z; k, m; a) - 1 &= (-1)^{k+1} L_m^k \cos \phi_m \\ &= (-1)^{k+1} L_m^k \cosh \Phi_m \\ &= (-1)^{k+1} L_m^k \cosh(H_m^k + B_m). \end{aligned} \quad (4.10)$$

Since $H_m^k \rightarrow H_k$, we have from (2.10) and (3.2)

$$f(z; k, m; a) - 1 \sim (-1)^{k+1} \frac{1}{\pi} \int_0^\infty \frac{e^{\zeta-t}}{t - \zeta} \left(\frac{t}{-\zeta} \right)^{k-\frac{1}{2}} dt.$$

Putting here $\zeta = -\lambda^2$ (see (4.9)) and $t = \mu^2$, we get (4.2). \square

5. UNWEIGHTED EXTREMAL POLYNOMIAL VIA CONFORMAL MAPPING

Let $P_m(z, p, a)$ be the best uniform (unweighted) approximation of $|x|^p$ by polynomials of degree less or equal $2m$, $2m > p$, on two intervals $[-1, -a] \cup [a, 1]$ and let $E = E_{2m}(p, a)$ be the approximation error.

In this section we prove

Theorem 5.1. *For a not even p there is a curve $\gamma = \gamma_m(p, a)$ inside the half-strip*

$$\{w = u + iv : u \in (0, (m+1)\pi), v > 0\} \quad (5.1)$$

such that the extremal polynomial possesses the representation

$$P_m(z, p, a) = z^p + (-1)^{[p/2]} E \cos \phi_m(z, p, a) \quad (5.2)$$

where $\phi_m(z, p, a)$ is the conformal map of the first quadrant onto the region $\Omega_m(p, a)$ in the half strip (5.1) bounded on the left by $\gamma_m(p, a)$. The conformal map is normalized by $\phi_m(a, p, a) = 0$, $\phi_m(1, p, a) = (m+1)\pi$ and $\phi_m(\infty, p, a) = \infty$. Moreover, the curve γ is the image of the imaginary half-axis under this conformal map that satisfies the following functional equation

$$\gamma_m(p, a) = \{u + iv = \phi_m(iy, p, a) : E \sin u(y) \sinh v(y) = \left| \sin \frac{\pi p}{2} \right| y^p, y > 0\}. \quad (5.3)$$

Proof. The proof contains two main ingredients: the Chebyshev theorem and the argument principle. In addition to that we will show some particular fact related to the shape of the extremal polynomial. We prove that $P_m(0, p, a) > E$ for even $[p/2]$ and $P_m(0, p, a) < -E$, when $[p/2]$ is odd.

Due to the symmetry of $P_m(x, p, a)$, we can use the Chebyshev theorem with respect to the best approximation of $(\sqrt{x})^p$ on $[a^2, 1]$ by polynomials of degree m . It gives us that $P_m(x, p, a)$ has $m+2$ points $\{x_j\}$ on the interval $[a, 1]$ where $P_m(x, p, a) - x^p$ alternates between $\pm E$ (the right half of the Chebyshev set in this case). Moreover, $x_0 = a$ and $x_{m+1} = 1$. From this remark we deduce that for $|t| < 1$ the equation

$$P_m(x, p, a) - x^p = tE \quad (5.4)$$

has precisely $m+1$ zeros, say $\{x_j(t)\}$, on $(a, 1)$. On the other hand

$$\{1, x^2, \dots, x^{2m}\} \cup \{x^p\}$$

forms the so called Chebyshev system on $[0, \infty)$, see e.g. [11, Ch. II, Sect. 2], and therefore (5.4) has no other solutions on the positive half axis ($m+1$ is the maximal possible number of roots for a generalized polynomial formed by a Chebyshev system of $m+2$ functions).

Using the argument principle we show that (5.4) has no other solutions in the whole quarter-plane.

Consider the contour that runs on the positive real axis till $x_j(t) - \epsilon$, then it goes around $x_j(t)$ on the half-circle of the radius ϵ clockwise. After the last of x_j 's we continue to go along the contour till the big positive R . Next piece of the contour is a quarter-circle till imaginary axis. Finally, from iR we go back to the origin. On each half-circle of the radius ϵ the argument of the function changes by $-\pi$. On the quarter circle it changes by about $\deg P_m(z, p, a) \times \frac{\pi}{2} = m\pi$.

It remains to show that the change of the argument on the last piece of the contour is about π . Then the whole change is $-(m+1)\pi + m\pi + \pi = 0$, and since the function has no poles, it has no zeros in the region.

Note that on the imaginary axis we have $\operatorname{Re}(P_m(iy, p, a) - (iy)^p) = P_m(iy, a) - \cos \frac{\pi p}{2} y^p$ and $\operatorname{Im}(P_m(iy, a) - (iy)^p) = -\sin \frac{\pi p}{2} y^p$. So the imaginary part increases with y for odd $\left[\frac{p}{2}\right]$ and decreases when it is even. Thus, it is enough to show that the real part changes from a certain negative value to $+\infty$ in the first case and, starting from a positive value for $y = 0$, it approaches to $-\infty$ as $y \rightarrow \infty$ in the second case (recall that $2m > p$).

We give here a self-contained proof of the above claim. For an alternative proof see Remark 5.2 below. Note that, if a is close to 1, for $p = 2k - 1$ the shape of the extremal unweighted polynomial is close to the shape of the extremal polynomial with the weight $|x|^{2k-1}$, see Remark 1.2. Consider, for example, the first case, $k - 1 = \left[\frac{p}{2}\right]$ is odd, then, due to Theorem 3.1 and the above remark,

$$P_m(a, p, a) - a^p = -E \quad \text{and} \quad P_m(1, p, a) - 1 = (-1)^m E. \quad (5.5)$$

Since $E_{2m}(p, a) \neq 0$ for all $0 < a < 1$, $2k - 2 < p < 2k$, no bifurcation is possible and relations (5.5) hold true for all values of (a, p) in the region. Since, moreover, (5.4) has no solutions on $\mathbb{R} \setminus [a, 1]$ we get the required behavior of $P_m(z, p, a) - z^p$ as $z = iy$, $y \rightarrow 0$ and $y \rightarrow +\infty$, from its behavior on the real axis $z = x$, as $x \rightarrow 0$ and $x \rightarrow +\infty$.

Thus $\arccos \frac{P_m(z, p, a) - z^p}{E}$ is well defined in the quarter-plane. We finish the proof by inspection of the boundary correspondence. \square

Note two facts: the curve (5.3) has the asymptote $u \rightarrow \pi$, $v \rightarrow +\infty$ ($y \rightarrow +\infty$) and we have uniqueness of the solution of the functional equation (5.3) due to uniqueness of the extremal polynomial.

Remark 5.2. Recall Gantmacher–Krein's Theorem (see e.g. [10, Theorem 4.4, more specifically Corollary 4.4]): the number of distinct zeros on $(0, \infty)$ of any generalized polynomial $\sum_{i=0}^n a_i x^{\alpha_i}$, where $\sum_{i=0}^n a_i^2 > 0$ and $\alpha_0, \alpha_1, \dots, \alpha_n$ is an increasing sequence of real numbers, is at most the number of sign changes in the sequence a_0, a_1, \dots, a_n after zero terms are discarded. Since the "polynomial" $P_m(x, p, a) - x^p - tE$, $-1 < t < 1$, has the maximal possible number of zeros in $(a, 1)$ its coefficients sequence (in the right order) has the maximal possible number of sign changes, that is $m + 1$. The coefficient before x^p is negative, therefore the signs of the zero coefficient, or

$P_m(0, p, a) - tE$, and the last one are $(-1)^{[\frac{p}{2}]}$ and $(-1)^{[\frac{p}{2}] + m + 1}$ respectively (compare (5.5)).

6. AND ITS ASYMPTOTICS

Theorem 6.1. *For the approximation error $E_{2m}(p, a)$ the limit (1.1) exists. Moreover, uniformly on compact subsets of the positive half-axis,*

$$\begin{aligned} \lim_{m \rightarrow \infty} \left\{ \left(\frac{m}{a} \right)^{\frac{p}{2}} P_m \left(\sqrt{\frac{a}{m}} \lambda, p, a \right) \right\} \\ = \lambda^p + \frac{\sin \frac{\pi p}{2}}{\pi} \int_0^\infty \mu^p e^{-(\lambda^2 + \mu^2)} \frac{2\mu d\mu}{\lambda^2 + \mu^2}. \end{aligned} \quad (6.1)$$

Proof. First we present briefly the second step of our method similar to the proof of Theorem 4.1. We use the representation (5.2). Then we use the symmetry principle and make convenient changes of variable to have a conformal map $\Phi_m(Z)$ of the upper plane in the region

$$i(\Omega_m(p, a) \cup \overline{\Omega_m(p, a)}) \cup (0, i\pi(m+1))$$

with the boundary correspondence

$$\Phi_m : (-C_m, -A_m, 0, A_m, C_m) \rightarrow (-\infty, -B_m, 0, B_m, \infty),$$

here $A_m = aC_m$ and $C_m = \frac{2m}{1-a^2}$.

For Φ_m we have the integral representation

$$\Phi_m(Z) = m \log \frac{1 + \frac{Z}{C_m}}{1 - \frac{Z}{C_m}} + \int_{A_m}^\infty \left[\frac{1}{X - Z} - \frac{1}{X + Z} \right] v_m(X) dX,$$

where

$$v_m(X) = \begin{cases} \frac{1}{\pi} \text{Im} \Phi_m(X), & A_m \leq X \leq C_m \\ 1, & X > C_m \end{cases} \quad (6.2)$$

and we put again

$$H_m(\zeta) = \Phi_m(Z) - B_m, \quad Z = A_m + \zeta.$$

Then

$$\begin{aligned} H_m(\zeta) &= m \log \frac{1 + a + \frac{\zeta}{C_m}}{1 - a - \frac{\zeta}{C_m}} + \int_0^\infty \left[\frac{1}{t - \zeta} - \frac{1}{t + 2A_m + \zeta} \right] \hat{v}_m(t) dt - B_m, \\ &\sim m \log \frac{1 + a}{1 - a} + \zeta + \int_0^\infty \left[\frac{1}{t - \zeta} - \frac{1}{t + 2A_m + \zeta} \right] \hat{v}_m(t) dt - B_m, \end{aligned} \quad (6.3)$$

In a usual way we write

$$\begin{aligned} & \int_0^\infty \left[\frac{1}{t-\zeta} - \frac{1}{t+2A_m+\zeta} \right] \hat{v}_m(t) dt \\ &= \int_0^\infty \left[\frac{1}{t-\zeta} - \frac{1}{t+2A_m+\zeta} \right] (\hat{v}_m(t) - \chi_{[1,\infty]}(t)) dt \\ & \quad - \log(1-\zeta) + \log(1+2A_m+\zeta) \end{aligned} \quad (6.4)$$

Since

$$H_m(\zeta) \rightarrow w(\zeta)$$

from (6.3), (6.4) we have

$$B_m - m \log \frac{1+a}{1-a} - \log(2A_m) \rightarrow -c \quad (6.5)$$

and

$$w(\zeta) = \zeta - \log(-\zeta) + c + \dots$$

as $\zeta \rightarrow \infty$.

A bit new element: rewrite the main equation (5.3) into the form (the right hand side is not a constant any more)

$$E_{2m} \operatorname{Im} \cosh \Phi_m = \left| \sin \frac{\pi p}{2} \right| y^p. \quad (6.6)$$

For $\zeta = \xi + i\eta$ we have

$$y = \sqrt{\frac{(A_m + \xi)^2 - A_m^2}{C_m^2 - (A_m + \xi)^2}} \sim \sqrt{\frac{2\xi a}{(1-a^2)C_m}} = \sqrt{\frac{\xi a}{m}}.$$

Thus (6.6) is of the form

$$\Lambda \operatorname{Im} e^w = \left| \sin \frac{\pi p}{2} \right| \xi^{\frac{p}{2}} \quad (6.7)$$

where

$$\Lambda = \lim_{m \rightarrow \infty} E_{2m} \left(\frac{m}{a} \right)^{\frac{p}{2}} \frac{1}{2} e^{B_m} \quad (6.8)$$

Finally, again, as the third step, we are looking for w in the form

$$w(\zeta) = \zeta + \log \left\{ \frac{1}{\pi} \int_0^\infty \frac{\tau(t) dt}{t-\zeta} \right\}. \quad (6.9)$$

Two small remarks on the normalization: due to $w(0) = 0$ we have

$$\frac{1}{\pi} \int_0^\infty \tau(t) \frac{dt}{t} = 1, \quad (6.10)$$

and also

$$e^c = \frac{1}{\pi} \int_0^\infty \tau(t) dt. \quad (6.11)$$

By representation (6.9) the main equation (6.7) is nothing but

$$\left(e^{\zeta \operatorname{Im} \frac{1}{\pi} \int_0^\infty \frac{\tau(t) dt}{t-\zeta}} \right)_{\zeta=\xi+i0} = \frac{\left| \sin \frac{\pi p}{2} \right|}{\Lambda} \xi^{\frac{p}{2}}. \quad (6.12)$$

Thus

$$\tau(\xi) = \frac{|\sin \frac{\pi p}{2}|}{\Lambda} \xi^{\frac{p}{2}} e^{-\xi} \quad (6.13)$$

and basically we are done. By (6.11)

$$e^c = \frac{|\sin \frac{\pi p}{2}|}{\pi \Lambda} \Gamma\left(\frac{p}{2} + 1\right) = \frac{1}{\Lambda |\Gamma(-\frac{p}{2})|}.$$

The constant Λ is uniquely defined by (6.10). By (6.8), (6.5) we have

$$E_{2m}(p, a) \sim 2\Lambda \left(\frac{a}{m}\right)^{\frac{p}{2}} \left(\frac{1-a}{1+a}\right)^m \frac{e^c}{2A_m} = \left(\frac{a}{m}\right)^{\frac{p}{2}} \left(\frac{1-a}{1+a}\right)^m \frac{1-a^2}{2am |\Gamma(-\frac{p}{2})|} \quad (6.14)$$

and (1.1) is proved.

The proof of (6.1) is similar to the proof of (4.2).

□

7. APPENDIX 1

Similar to (5.2) for extremal entire function F and for $a = 0$ we write

$$F(z) = z^p + (-1)^{[\frac{p}{2}]} E \cos \phi(z), \quad (7.1)$$

or for $z = -i\lambda$ and $\phi = -i\psi$

$$F(i\lambda) = (-i\lambda)^p + (-1)^{[\frac{p}{2}]} E \cosh \psi(\lambda). \quad (7.2)$$

Now, $\psi(\lambda)$ is the conformal map of the upper half-plane on the upper half-plane

$$\psi(\lambda) = \lambda + \log \left\{ C + \frac{1}{\pi} \int \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1 + \mu^2} \right) \rho(\mu) d\mu \right\}. \quad (7.3)$$

Note that in this representation ρ is *not* symmetric.

Put

$$C + \frac{1}{\pi} \int \left(\frac{1}{\mu - \lambda} - \frac{\mu}{1 + \mu^2} \right) \rho(\mu) d\mu = -\tilde{\rho} + i\rho,$$

where $\tilde{\rho}$ is "a kind of Hilbert transform" of ρ . Then we get from (7.2)

$$F(i\lambda) = e^{-i\frac{\pi p}{2}} \lambda^p + (-1)^{[\frac{p}{2}]} \frac{E}{2} \left\{ e^\lambda (-\tilde{\rho} + i\rho) + \frac{e^{-\lambda}}{-\tilde{\rho} + i\rho} \right\}. \quad (7.4)$$

For real λ 's the imaginary part of this expression gives us

$$0 = -\sin \frac{\pi p}{2} \lambda^p + (-1)^{[\frac{p}{2}]} \frac{E}{2} \left\{ e^\lambda \rho - \frac{e^{-\lambda} \rho}{\tilde{\rho}^2 + \rho^2} \right\}. \quad (7.5)$$

Conjecture 7.1. For the Bernstein Problem, $a = 0$, we conjecture that the extremal entire function F is of the form (7.1), where ϕ is the conformal map of the upper half plane onto the region in the upper half plane above the curve

$$\gamma = \{u + iv = \phi(x) : x \in \mathbb{R}\}$$

such that

$$L \sin v(x) \sinh u(x) = x, \quad x \in \mathbb{R} \quad (7.6)$$

and normalized by $\phi(0) = 0$, $\phi(z) \sim z$, $z \rightarrow \infty$.

Let us rewrite the above equation in terms of the unknown function, say ρ and its Hilbert transform $\tilde{\rho}$. We use the integral representation

$$\phi(z) = z + \frac{1}{\pi} \int_0^\infty \left[\frac{1}{x-z} - \frac{1}{x+z} \right] v(x) dx.$$

The curve has the asymptote $v \rightarrow \pi$, $x \rightarrow \infty$. We define $\rho := \pi - v$ to write

$$\phi(z) = z + i\pi - \frac{1}{\pi} \int_0^\infty \frac{\rho dx}{x-z}.$$

Finally since

$$\frac{1}{\pi} \int_0^\infty \frac{\rho dx}{x-z} = -\tilde{\rho} + i\rho,$$

we get $u(x) = \tilde{\rho}(x) + x$ and $v(x) = \pi - \rho(x)$. Thus equation (7.6) leads to

$$L \sin \rho(x) \sinh(\tilde{\rho}(x) + x) = x. \quad (7.7)$$

8. APPENDIX 2

From [1], problem 42:

$$E_l \left[\frac{1}{(b+x)^s} \right] \sim \frac{l^{s-1}}{|\Gamma(s)|} \frac{(b - \sqrt{b^2 - 1})^l}{(b^2 - 1)^{\frac{s+1}{2}}} \quad (b > 1, s \neq 0), \quad (8.1)$$

where $E_l[f(x)]$ is the error of the approximation of the function $f(x)$ on the interval $[-1, 1]$ by polynomials of degree not more than l .

We change the variable

$$y = \frac{b+x}{b+1}$$

and put $a^2 = \frac{b-1}{b+1}$. Then we have

$$\inf_{P: \deg P \leq l} \max_{y \in [a^2, 1]} |y^{-s} - P(y)| = (1+b)^s E_l \left[\frac{1}{(b+x)^s} \right].$$

That is

$$E_{2l}(-2s, a) = (1+b)^s E_l \left[\frac{1}{(b+x)^s} \right]. \quad (8.2)$$

Note that

$$b = \frac{1+a^2}{1-a^2}, \quad b^2 - 1 = \frac{4a^2}{(1-a^2)^2},$$

and therefore

$$\sqrt{b^2 - 1} = \frac{2a}{1-a^2}, \quad b - \sqrt{b^2 - 1} = \frac{1-a}{1+a}.$$

Thus from (8.1) and (8.2) we get

$$\begin{aligned}
 E_{2l}(-2s, a) &\sim \left(\frac{2}{1-a^2}\right)^s \frac{l^{s-1}}{|\Gamma(s)|} \left(\frac{1-a}{1+a}\right)^l \left(\frac{1-a^2}{2a}\right)^{s+1} \\
 &= a^{-s} \frac{l^{s-1}}{|\Gamma(s)|} \left(\frac{1-a}{1+a}\right)^l \left(\frac{1-a^2}{2a}\right) \\
 &= a^{-s-1} \frac{l^{s-1}}{|\Gamma(s)|} \left(\frac{1-a}{1+a}\right)^{l+1} \frac{(1+a)^2}{2}.
 \end{aligned}$$

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